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Non-Noether symmetries and their influence on phase space geometry

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Abstract

In the present paper some geometric aspects of the concept of non-Noether symmetry are discussed. It is shown that in regular Hamiltonian systems such a symmetry canonically leads to a Lax pair on the algebra of linear operators on cotangent bundle over the phase space. Correspondence between the non-Noether symmetries and other wide spread geometric methods of generating conservation laws such as bi-Hamiltonian formalism, bidifferential calculi and Frölicher–Nijenhuis geometry is considered. It is proved that the integrals of motion associated with the continuous non-Noether symmetry are in involution whenever the generator of the symmetry satisfies a certain Yang–Baxter type equation.

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Recently non-Noether symmetries were actively discussed by several authors [3,9,11,13] and some new interesting results has been obtained. Here we would like to shed more light on geometric aspects of the concept of non-Noether symmetry and to emphasize influence of such a symmetries on the phase space geometry. Partially the motivation for studying these issues comes from the theory of integrable models that essentially relies on different geometric structures used in construction of the conservation laws and the invariant Lagrangian submanifolds. Among them are Frölicher–Nijenhuis operators, bi-Hamiltonian systems, Lax pairs and bicomplexes. Unfortunately these important structures carry no

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direct physical content and are considered as a purely mathematical constructions resulting the conservation laws, but it seems that they could be related to the symmetries of the dynamical systems. In the present paper we would like to show that in Hamiltonian systems presence of certain non-Noether symmetries canonically leads to the above mentioned Lax pairs, Frölicher–Nijenhuis operators, bi-Hamiltonian structures, bicomplexes and a number of conservation laws.

We first recall some basic knowledge of the Hamiltonian dynamics. The phase space of a regular Hamiltonian system is a Poisson manifold—a smooth finite-dimensional manifold equipped with the Poisson bivector field W subjected to the following condition:

$$[W, W] = 0, \tag{1}$$

where square bracket stands for Schouten bracket or supercommutator (for simplicity further it will be referred as commutator). In a standard manner Poisson bivector field defines a Lie bracket on the algebra of observables (smooth real-valued functions on phase space) called Poisson bracket:

$$\{f, g\} = W(\mathrm{d}f \wedge \mathrm{d}g) = L_{W(f)}g = -L_{W(g)}f, \tag{2}$$

where $W(f)$ and $W(g)$ are Hamiltonian vector fields associated with the functions f and g, respectively, while L denotes Lie derivative. Skew symmetry of the bivector field W provides the skew symmetry of the corresponding Poisson bracket and the condition (1) ensures that for every triple (f, g, h) of smooth functions on the phase space the Jacobi identity

$$\{f\{g, h\}\} + \{h\{f, g\}\} + \{g\{h, f\}\} = 0 \tag{3}$$

is satisfied. We also assume that the dynamical system under consideration is regular—the bivector field W has maximal rank, i.e. its nth outer power, where n is a half-dimension of the phase space, does not vanish $W^n \neq 0$. In this case W gives rise to a well known isomorphism Φ_W between the differential 1-forms and the vector fields defined by

$$\Phi_W(u) = W(u) \tag{4}$$

for every 1-form u.

Time evolution of observables is governed by the Hamilton's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \{h, f\}, \tag{5}$$

where h is some fixed smooth function on the phase space called Hamiltonian. Let us recall that each vector field E on the phase space generates the one-parameter continuous group of transformations (flow) $g_a = \mathrm{e}^{aL_E}$ that acts on the observables as follows:

$$g_a(f) = \mathrm{e}^{aL_E}(f) = f + aL_E f + \tfrac{1}{2}a^2 L_E^2 f + \cdots . \tag{6}$$

Such a group of transformation is called symmetry of Hamilton's equation (5) if it commutes with time evolution operator

$$\frac{\mathrm{d}}{\mathrm{d}t}g_a(f) = g_a\left(\frac{\mathrm{d}}{\mathrm{d}t}f\right) \tag{7}$$

or

$$L_{W(h)}g_a(f) = g_a(L_{W(h)}f) \tag{8}$$

that is possible only if the generator E of the group commutes with $W(h) = \{h, \cdot\}$, i.e. if

$$[E, W(h)] = 0. \tag{9}$$

Symmetry is said to be Noether if its generator is Hamiltonian vector field (by its definition every Hamiltonian vector field E could be represented in the form $E = W(f) = \{f, \cdot\}$ for some smooth function f on phase space and, according to Liouville's theorem, such a vector fields preserve Poisson bivector, $[E, W] = 0$) and non-Noether whenever its generator is non-Hamiltonian, $[E, W] \neq 0$. Let us focus on non-Noether symmetries. We would like to show that the presence of such a symmetry could essentially enrich the geometry of the phase space and under the certain conditions could ensure integrability of the dynamical system. Before we proceed let us recall that the non-Noether symmetry leads to a number of integrals of motion [13] (see also [2,3,9,11,12]). More precisely the relationship between non-Noether symmetries and the conservation laws is described by the following theorem.

Theorem 1. *Let (M, h) be a regular Hamiltonian system on the $2n$-dimensional Poisson manifold M. Then, if the vector field E generates non-Noether symmetry, the functions*

$$Y^{(k)} = \frac{\hat{W}^k \wedge W^{n-k}}{W^n}, \quad k = 1, 2, \ldots, n, \tag{10}$$

where $\hat{W} = [E, W]$, are integrals of motion.

Proof. By the definition

$$\hat{W}^k \wedge W^{n-k} = Y^{(k)} W^n. \tag{11}$$

Let us take the Lie derivative of this expression along the vector field $W(h)$,

$$[W(h), \hat{W}^k \wedge W^{n-k}] = \frac{\mathrm{d}}{\mathrm{d}t}(Y^{(k)})W^n + Y^{(k)}[W(h), W^n] \tag{12}$$

or

$$\begin{aligned} &k[W(h), \hat{W}] \wedge \hat{W}^{k-1} \wedge W^{n-k} + (n-k)[W(h), W] \wedge \hat{W}^k \wedge W^{n-k-1} \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}(Y^{(k)})W^n + nY^{(k)}[W(h), W] \wedge W^{n-1}, \end{aligned} \tag{13}$$

but according to the Liouville theorem the Hamiltonian vector field preserves W, i.e.

$$[W(h), W] = 0, \tag{14}$$

hence, by taking into account that $[W(h), E] = 0$ we get

$$[W(h), \hat{W}] = [W(h)[E, W]] = [W[W(h), E]] + [E[W, W(h)]] = 0, \tag{15}$$

and as a result (13) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}Y^{(k)}W^n = 0, \tag{16}$$

but since the dynamical system is regular ($W^n \neq 0$) we obtain that the functions $Y^{(k)}$ are integrals of motion. □

Remark 1. Instead of conserved quantities $Y^{(1)} \cdots Y^{(n)}$, the solutions $c_1 \cdots c_n$ of the secular equation

$$(\hat{W} - cW)^n = 0 \tag{17}$$

could be associated with the generator of symmetry. By expanding expression (17) it is easy to verify that the integrals of motion $Y^{(k)}$ can be expressed in terms of the functions $c_1 \cdots c_n$ in the following form:

$$Y^{(k)} = \frac{(n-k)!k!}{n!} \sum_{i_p \neq i_s} c_{i_1} c_{i_2} \cdots c_{i_k}. \tag{18}$$

Presence of the non-Noether symmetry not only leads to a sequence of conservation laws, but also endows the phase space with a number of interesting geometric structures and it appears that such a symmetry is related to many important concepts used in theory of dynamical systems. One of the such concepts is Lax pair. Let us recall that Lax pair of Hamiltonian system on Poisson manifold M is a pair (L, P) of smooth functions on M with values in some Lie algebra g such that the time evolution of L is governed by the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [L, P], \tag{19}$$

where $[\cdot, \cdot]$ is a Lie bracket on g. It is well known that each Lax pair leads to a number of conservation laws. When g is some matrix Lie algebra the conservation laws are just traces of powers of L

$$I^{(k)} = \mathrm{Tr}(L^k). \tag{20}$$

It is remarkable that each generator of the non-Noether symmetry canonically leads to the Lax pair of a certain type. In the local coordinates z_a, where the bivector field W and the generator of the symmetry E have the following form:

$$W = \sum_{ab} W_{ab} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b}, \qquad E = \sum_a E_a \frac{\partial}{\partial z_a} \tag{21}$$

corresponding Lax pair could be calculated explicitly. Namely we have the following theorem.

Theorem 2. *Let (M, h) be a regular Hamiltonian system on the $2n$-dimensional Poisson manifold M. Then, if the vector field E on M generates the non-Noether symmetry, the following $2n \times 2n$ matrix-valued functions on M*:

$$L_{ab} = \sum_{dc} (W^{-1})_{ad} \left(E_c \frac{\partial W_{db}}{\partial z_c} - W_{cb} \frac{\partial E_d}{\partial z_c} + W_{dc} \frac{\partial E_b}{\partial z_c} \right), \tag{22}$$

$$P_{ab} = \sum_c \left(\frac{\partial W_{bc}}{\partial z_a} \frac{\partial h}{\partial z_c} + W_{bc} \frac{\partial^2 h}{\partial z_a \partial z_c} \right), \tag{23}$$

form the Lax pair (19) of the dynamical system (M, h).

Proof. Let us consider the following operator on a space of 1-forms:

$$\bar{R}_E(u) = \Phi_W^{-1}([E, \Phi_W(u)]) - L_E u \tag{24}$$

(here Φ_W is the isomorphism (4)). It is obvious that $\bar{R}_E$ is a linear operator and it is invariant since the evolution operator $W(h)$ commutes with both Φ_W (as far as $[W(h), W] = 0$) and E (because E generates symmetry $[E, W(h)] = 0$). In the terms of the local coordinates $\bar{R}_E$ has the following form:

$$\bar{R}_E = \sum_{ab} L_{ab}\, \mathrm{d}z_a \otimes \frac{\partial}{\partial z_b} \tag{25}$$

and the invariance condition

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{R}_E = L_{W(h)} \bar{R}_E = 0 \tag{26}$$

yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \bar{R}_E &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{ab} L_{ab}\, \mathrm{d}z_a \otimes \frac{\partial}{\partial z_b} \\
&= \sum_{ab} \left(\frac{\mathrm{d}}{\mathrm{d}t} L_{ab} \right) \mathrm{d}z_a \otimes \frac{\partial}{\partial z_b} + \sum_{ab} L_{ab} (L_{W(h)}\, \mathrm{d}z_a) \otimes \frac{\partial}{\partial z_b} \\
&\quad + \sum_{ab} L_{ab}\, \mathrm{d}z_a \otimes \left(L_{W(h)} \frac{\partial}{\partial z_b} \right) \\
&= \sum_{ab} \left(\frac{\mathrm{d}}{\mathrm{d}t} L_{ab} \right) \mathrm{d}z_a \otimes \frac{\partial}{\partial z_b} + \sum_{abcd} L_{ab} \partial_c (W_{ad} \partial_d h)\, \mathrm{d}z_c \otimes \frac{\partial}{\partial z_b} \\
&\quad + \sum_{abcd} L_{ab} \partial_b (W_{cd} \partial_d h)\, \mathrm{d}z_a \otimes \frac{\partial}{\partial z_c} \\
&= \sum_{ab} \left(\frac{\mathrm{d}}{\mathrm{d}t} L_{ab} + \sum_c (P_{ac} L_{cb} - L_{ac} P_{cb}) \right) \mathrm{d}z_a \otimes \frac{\partial}{\partial z_b} = 0
\end{aligned} \tag{27}$$

or in matrix notations

$$\frac{\mathrm{d}}{\mathrm{d}t} L = [L, P]. \tag{28}$$

So, we have proved that the non-Noether symmetry canonically yields a Lax pair on the algebra of linear operators on cotangent bundle over the phase space. □

Remark 2. The conservation laws (20) associated with the Lax pair (28) can be expressed in terms of the integrals of motion c_i in quite simple way:

$$I^{(k)} = \mathrm{Tr}(L^k) = \sum_i c_i^k. \tag{29}$$

This correspondence follows from Eq. (17) and the definition of the operator R_E (24).

Now let us focus on the integrability issues. We know that n integrals of motion are associated with each generator of non-Noether symmetry and according to the Liouville–Arnold theorem Hamiltonian system is completely integrable if it possesses n functionally independent integrals of motion in involution (two functions f and g are said to be in involution if their Poisson bracket vanishes $\{f, g\} = 0$). Generally speaking the conservation laws associated with symmetry might appear to be neither independent nor involutive. However, it is reasonable to ask the question—what condition should be satisfied by the generator of the symmetry to ensure the involutivity ($\{Y^{(k)}, Y^{(m)}\} = 0$) of conserved quantities? In Lax theory such a condition is known as classical Yang–Baxter equation (CYBE). Since involutivity of the conservation laws is closely related to the integrability it is essential to have some analog of CYBE for the generator of non-Noether symmetry. To address this issue we would like to propose the following theorem.

Theorem 3. *If the vector field E on* $2n$*-dimensional Poisson manifold M satisfies the condition*

$$[[E[E, W]]W] = 0 \tag{30}$$

and W bivector field has maximal rank ($W^n \neq 0$) *then the functions* (10) *are in involution*

$$\{Y^{(k)}, Y^{(m)}\} = 0. \tag{31}$$

Proof. First of all let us note that the identity (1) satisfied by the Poisson bivector field W is responsible for the Liouville theorem

$$[W, W] = 0 \Leftrightarrow L_{W(f)}W = [W(f), W] = 0. \tag{32}$$

By taking the Lie derivative of the expression (1) we obtain another useful identity

$$L_E[W, W] = [E[W, W]] = [[E, W]W] + [W[E, W]] = 2[\hat{W}, W] = 0. \tag{33}$$

This identity gives rise to the following relation:

$$[\hat{W}, W] = 0 \Leftrightarrow [\hat{W}(f), W] = -[\hat{W}, W(f)] \tag{34}$$

and finally condition (30) ensures third identity

$$[\hat{W}, \hat{W}] = 0 \tag{35}$$

yielding Liouville theorem for $\hat{W}$

$$[\hat{W}, \hat{W}] = 0 \Leftrightarrow [\hat{W}(f), \hat{W}] = 0. \tag{36}$$

Indeed

$$[\hat{W}, \hat{W}] = [[E, W]\hat{W}] = [[\hat{W}, E]W] = -[[E, \hat{W}]W] = -[[E[E, W]]W] = 0. \quad (37)$$

Now let us consider two different solutions $c_i \neq c_j$ of Eq. (17). By taking the Lie derivative of the equation

$$(\hat{W} - c_i W)^n = 0 \quad (38)$$

along the vector fields $W(c_j)$ and $\hat{W}(c_j)$ and using Liouville theorem for W and $\hat{W}$ bivectors we obtain the following relations:

$$(\hat{W} - c_i W)^{n-1}(L_{W(c_j)}\hat{W} - \{c_j, c_i\}W) = 0 \quad (39)$$

and

$$(\hat{W} - c_i W)^{n-1}(c_i L_{\hat{W}(c_j)} W + \{c_j, c_i\}_\bullet W) = 0, \quad (40)$$

where

$$\{c_i, c_j\}_\bullet = \hat{W}(\mathrm{d}c_i \wedge \mathrm{d}c_j) \quad (41)$$

is the Poisson bracket calculated by means of the bivector field $\hat{W}$. Now multiplying (39) by c_i subtracting (40) and using identity (34) gives rise to

$$(\{c_i, c_j\}_\bullet - c_j\{c_i, c_j\})(\hat{W} - c_i W)^{n-1} W = 0. \quad (42)$$

Thus, either

$$\{c_i, c_j\}_\bullet - c_j\{c_i, c_j\} = 0 \quad (43)$$

or the volume field $(\hat{W} - c_i W)^{n-1} W$ vanishes. In the second case we can repeat (39)–(42) procedure for the volume field $(\hat{W} - c_i W)^{n-1} W$ yielding after n iterations $W^n = 0$ that according to our assumption (that the dynamical system is regular) is not true. As a result we arrived at (43) and by the simple interchange of indices $i \leftrightarrow j$ we get

$$\{c_i, c_j\}_\bullet - c_i\{c_i, c_j\} = 0. \quad (44)$$

Finally by comparing (43) and (44) we obtain that the functions c_i are in involution with respect to the both Poisson structures (since $c_i \neq c_j$)

$$\{c_i, c_j\}_\bullet = \{c_i, c_j\} = 0 \quad (45)$$

and according to (18) the same is true for the integrals of motion $Y^{(k)}$. □

Corollary 1. *Each generator of non-Noether symmetry satisfying* Eq. (30) *endows dynamical system with the bi-Hamiltonian structure* [1,3,4,8,10]*—couple* $(W, \hat{W})$ *of compatible* $([W, \hat{W}] = 0)$ *Poisson* $([W, W] = [\hat{W}, \hat{W}] = 0)$ *bivector fields.*

It often happens that n conservation laws associated with the generator of non-Noether symmetry appear to be functionally independent, involutive and ensure the integrability of the dynamical system. Such examples are especially interesting in the case of infinite-dimensional Hamiltonian systems, where a single generator of non-Noether symmetry yields infinite number of integrals of motion (note that, regardless the dimension of the phase space, the generator of Noether symmetry reproduces only one conservation law). Let us consider well known infinite-dimensional Hamiltonian system—Korteweg–de Vries equation.

Example 1. KdV equation not only possesses infinite number of conservation laws in involution, but also could be endowed with bi-Hamiltonian structure [14], admits Lax pair formulation and has many interesting geometric properties. Let us recall that the KdV equation

$$u_t + u_{xxx} + uu_x = 0 \tag{46}$$

(here $u = u(t, x)$ is a smooth real-valued function subjected to the boundary conditions $u(t, \pm\infty) = 0$) is a Hamilton's equation (5) on infinite-dimensional Poisson manifold with corresponding Hamiltonian equal to

$$h = \int_{-\infty}^{+\infty} \left(\frac{u^3}{3} - u_x^2\right) \mathrm{d}x \tag{47}$$

and Poisson bivector field

$$W = \int_{-\infty}^{+\infty} \mathrm{d}x \frac{\delta}{\delta u} \wedge \frac{\delta}{\delta v}, \quad v = \int_{-\infty}^{x} u(\xi)\, \mathrm{d}\xi, \tag{48}$$

where $\delta/\delta u$ denotes variational derivative with respect to u. Observables are smooth functionals of u and its spatial derivatives u_x, u_{xx}, etc. Each vector field E generates infinitisimal transformation g_ϵ of algebra of observables defined, for every functional F, by the equation

$$g_\epsilon(F) = 1 + \epsilon E(F) + \mathrm{O}(\epsilon). \tag{49}$$

Like in the finite-dimensional case such a transformation is called symmetry of the Hamilton's equation if it commutes with time evolution operator

$$g_\epsilon \left(\frac{\mathrm{d}}{\mathrm{d}t} F\right) = \frac{\mathrm{d}}{\mathrm{d}t} g_\epsilon(F). \tag{50}$$

Quite recently it appeared that KdV equation possesses hidden non-Noether symmetry [3] generated by the vector field

$$E = \int_{-\infty}^{+\infty} \mathrm{d}x (u_{xx} + \tfrac{1}{3}u^2) \frac{\delta}{\delta u} + X_{\mathrm{H}} \tag{51}$$

(where X_{H} is the Hamiltonian part of symmetry generator, see [3]). This symmetry gives rise to the well known infinite sequence of conservation laws of KdV equation

$$
\begin{aligned}
&I^{(1)} = \int_{-\infty}^{+\infty} u\,\mathrm{d}x, \qquad I^{(2)} = \int_{-\infty}^{+\infty} u^2\,\mathrm{d}x, \qquad I^{(3)} = \int_{-\infty}^{+\infty} \left(\frac{u^3}{3} - u_x^2\right)\mathrm{d}x,\\
&I^{(4)} = \int_{-\infty}^{+\infty} \left(\frac{5}{36}u^4 - \frac{5}{3}uu_x^2 + u_{xx}^2\right)\mathrm{d}x,\\
&I^{(5)} = \int_{-\infty}^{+\infty} \left(\frac{7}{108}u^5 - \frac{35}{18}u^2u_x^2 + \frac{7}{3}uu_{xx}^2 - u_{xxx}^2\right)\mathrm{d}x,\\
&I^{(6)} = \int_{-\infty}^{+\infty} \left(\frac{7}{216}u^6 - \frac{35}{18}u^3u_x^2 + \frac{7}{2}u^2u_{xx}^2 - \frac{35}{36}u_x^4 - 3uu_{xxx}^2 + \frac{10}{3}u_{xx}^3 - u_{xxxx}^2\right)\mathrm{d}x.
\end{aligned}
\tag{52}
$$

It is easy to verify that the vector field E satisfies condition (30) and due to Theorem 3 (strictly speaking the Poisson manifold in Theorems 1–3 is finite dimensional and extension of the main results to the infinite-dimensional case requires additional explanation and justification that lays outside the scope of present article) the conservation laws of KdV equation are in involution. Functional independence of conservation laws is obvious so in this case the symmetry is responsible for integrability. According to Theorem 2 this symmetry canonically leads to a Lax pair, that appears to be known as KdV Lax pair:

$$L = \partial_x^2 + \tfrac{1}{6}u, \tag{53}$$

$$P = \partial_x^3 + \tfrac{1}{6}(u\partial_x + u_x), \tag{54}$$

while the operator $\bar{R}_E$ exactly reproduces Lenard recursion operator

$$\bar{R}_E = \int_{-\infty}^{+\infty} \mathrm{d}x \left(\delta u_{xx} \otimes \frac{\delta}{\delta u} + \frac{2}{3}u\delta u \otimes \frac{\delta}{\delta u}\right). \tag{55}$$

Due to corollary of Theorem 3 the phase space of KdV equation is endowed with a bi-Hamiltonian structure [3] that coincides with the one discovered by Magri [14].

Another concept that is often used in theory of dynamical systems and could be related to the non-Noether symmetry is the bidifferential calculus (bicomplex approach). Recently Dimakis and Müller-Hoissen applied bidifferential calculi to the wide range of integrable models including KdV hierarchy, KP equation, self-dual Yang–Mills equation, Sine–Gordon equation, Toda models, and non-linear Schrödinger and Liouville equations. It turns out that these models can be effectively described and analyzed using the bidifferential calculi [4,6,7,10].

Under the bidifferential calculus we mean the graded algebra of differential forms

$$\Omega = \bigcup_{k=0}^{\infty} \Omega^{(k)} \tag{56}$$

($\Omega^{(k)}$ denotes the space of k-degree differential forms) equipped with a couple of differential operators

$$\mathrm{d}, \tilde{\mathrm{d}} : \Omega^{(k)} \to \Omega^{(k+1)} \tag{57}$$

satisfying $\mathrm{d}^2 = \tilde{\mathrm{d}}^2 = \mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d} = 0$ conditions (see [6,7]). It is interesting that if generator of the non-Noether symmetry satisfies Eq. (30) then we are able to construct an invariant bidifferential calculus of a certain type. This construction is summarized in the following theorem.

Theorem 4. *Let (M, h) be a regular Hamiltonian system on the Poisson manifold M. Then, if the vector field E on M generates the non-Noether symmetry and satisfies* Eq. (30)*, the differential operators*

$$\mathrm{d}u = \Phi_W^{-1}([W, \Phi_W(u)]), \tag{58}$$

$$\tilde{\mathrm{d}}u = \Phi_W^{-1}([[E, W]\Phi_W(u)]) \tag{59}$$

form invariant bidifferential calculus ($\mathrm{d}^2 = \tilde{\mathrm{d}}^2 = \mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d} = 0$) *over the graded algebra of differential forms on M.*

Proof. First of all we have to show that d and $\tilde{\mathrm{d}}$ are really differential operators, i.e. they are linear maps from $\Omega^{(k)}$ into $\Omega^{(k+1)}$, satisfy derivation property and are nilpotent ($\mathrm{d}^2 = \tilde{\mathrm{d}}^2 = 0$). Linearity is obvious and follows from the linearity of the Schouten bracket $[\cdot, \cdot]$ and Φ_W, Φ_W^{-1} maps. Then, if u is a k-degree form Φ_W maps it on k-degree multivector field and the Schouten brackets $[W, \Phi_W(u)]$ and $[[E, W]\Phi_W(u)]$ result the $(k+1)$-degree multivector fields that are mapped to $(k+1)$-degree differential forms by Φ_W^{-1}. So, d and $\tilde{\mathrm{d}}$ are linear maps from $\Omega^{(k)}$ into $\Omega^{(k+1)}$. Derivation property follows from the same feature of the Schouten bracket $[\cdot, \cdot]$ and linearity of Φ_W and Φ_W^{-1} maps. Now we have to prove the nilpotency of d and $\tilde{\mathrm{d}}$. Let us consider $\mathrm{d}^2 u$

$$\mathrm{d}^2 u = \Phi_W^{-1}([W, \Phi_W(\Phi_W^{-1}([W, \Phi_W(u)]))]) = \Phi_W^{-1}([W[W, \Phi_W(u)]]) = 0 \tag{60}$$

as a result of the property (32) and the Jacobi identity for $[\cdot, \cdot]$ bracket. In the same manner

$$\tilde{\mathrm{d}}^2 u = \Phi_W^{-1}([[W, E][[W, E]\Phi_W(u)]]) = 0 \tag{61}$$

according to the property (36) of $[W, E] = \hat{W}$ and the Jacobi identity. Thus, we have proved that d and $\tilde{\mathrm{d}}$ are differential operators (in fact d is ordinary exterior differential and the expression (58) is its well known representation in terms of Poisson bivector field). It remains to show that the compatibility condition $\mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d} = 0$ is fulfilled. Using definitions of d, $\tilde{\mathrm{d}}$ and the Jacobi identity we get

$$(\mathrm{d}\tilde{\mathrm{d}} + \tilde{\mathrm{d}}\mathrm{d})(u) = \Phi_W^{-1}([[[W, E]W]\Phi_W(u)]) = 0 \tag{62}$$

as far as (34) is satisfied. So, d and $\tilde{\mathrm{d}}$ form the bidifferential calculus over the graded algebra of differential forms. It is also clear that the bidifferential calculus d, $\tilde{\mathrm{d}}$ is invariant, since both d and $\tilde{\mathrm{d}}$ commute with time evolution operator $W(h) = \{h, \cdot\}$. □

Remark 3. Conservation laws that are associated with the bidifferential calculus (58) and (59) and form Lenard scheme (see [6,7]):

$$\tilde{\mathrm{d}}I^{(k)} = \mathrm{d}I^{(k+1)} \tag{63}$$

coincide with the sequence of integrals of motion (29). Proof of this correspondence lay outside the scope of present article, but could be done in the manner similar to [4].

Finally we would like to reveal some features of the operator $\bar{R}_E$ (24) and to show how Frölicher–Nijenhuis geometry could arise in Hamiltonian system that possesses certain non-Noether symmetry. From the geometric properties of the tangent-valued forms we know [5] that the traces of powers of a linear operator F on tangent bundle are in involution whenever its Frölicher–Nijenhuis torsion $T(F)$ vanishes, i.e. whenever for arbitrary vector fields X, Y the condition

$$T(F)(X,Y) = [FX, FY] - F([FX,Y] + [X,FY] - F[X,Y]) = 0 \tag{64}$$

is satisfied. Torsionless forms are also called Frölicher–Nijenhuis operators and are widely used in theory of integrable models [5]. We would like to show that each generator of non-Noether symmetry satisfying Eq. (30) canonically leads to invariant Frölicher–Nijenhuis operator on tangent bundle over the phase space. Strictly speaking we have the following theorem.

Theorem 5. *Let (M, h) be a regular Hamiltonian system on the Poisson manifold M. If the vector field E on M generates the non-Noether symmetry and satisfies* Eq. (30) *then the linear operator, defined for every vector field X by equation*

$$R_E(X) = \Phi_W(L_E \Phi_W^{-1}(X)) - [E, X] \tag{65}$$

is invariant Frölicher–Nijenhuis operator on M.

Proof. Invariance of R_E follows from the invariance of the $\bar{R}_E$ defined by (24) (note that for arbitrary 1-form vector field u and vector field X contraction $i_X u$ has the property $i_{R_E X} u = i_X \bar{R}_E u$, so R_E is actually transposed to $\bar{R}_E$). It remains to show that the condition (30) ensures vanishing of the Frölicher–Nijenhuis torsion $T(R_E)$ of R_E, i.e. for arbitrary vector fields X, Y

$$\begin{aligned} T(R_E)(X,Y) &= [R_E(X), R_E(Y)] - R_E([R_E(X), Y] \\ &\quad + [X, R_E(Y)] - R_E([X,Y])) = 0. \end{aligned} \tag{66}$$

First let us introduce the following auxiliary 2-forms:

$$\omega = \Phi_W^{-1}(W), \qquad \omega^{\bullet} = \bar{R}_E \omega, \qquad \omega^{\bullet\bullet} = \bar{R}_E \omega^{\bullet}. \tag{67}$$

Using the realization (58) of the differential d and the property (1) yields

$$\mathrm{d}\omega = \Phi_W^{-1}([W, W]) = 0. \tag{68}$$

Similarly, using the property (34) we obtain

$$\mathrm{d}\omega^{\bullet} = \mathrm{d}\Phi_W^{-1}([E, W]) - \mathrm{d}L_E\omega = \Phi_W^{-1}([[E,W]W]) - L_E\,\mathrm{d}\omega = 0. \tag{69}$$

And finally, taking into account that $\omega^{\bullet} = 2\Phi_W^{-1}([E, W])$ and using the condition (30), we get

$$\mathrm{d}\omega^{\bullet\bullet} = 2\Phi_W^{-1}([[E[E,W]]W]) - 2\,\mathrm{d}L_E\omega^{\bullet} = -2L_E\,\mathrm{d}\omega^{\bullet} = 0. \tag{70}$$

So the differential forms ω, $\omega^\bullet$, $\omega^{\bullet\bullet}$ are closed

$$\mathrm{d}\omega = \mathrm{d}\omega^\bullet = \mathrm{d}\omega^{\bullet\bullet} = 0. \tag{71}$$

Now let us consider the contraction of $T(R_E)(X, Y)$ and ω.

$$\begin{aligned} i_{T(R_E)(X,Y)}\omega &= i_{[R_E X, R_E Y]}\omega - i_{[R_E X, Y]}\omega^\bullet - i_{[X, R_E Y]}\omega^\bullet + i_{[X,Y]}\omega^{\bullet\bullet} \\ &= L_{R_E X} i_Y \omega^\bullet - i_{R_E Y} L_X \omega^\bullet - L_{R_E X} i_Y \omega^\bullet + i_Y L_{R_E X}\omega^\bullet \\ &\quad - L_X i_{R_E Y}\omega^\bullet + i_{R_E Y} L_X \omega^\bullet + i_{[X,Y]}\omega^{\bullet\bullet} \\ &= i_Y L_X \omega^{\bullet\bullet} - L_X i_Y \omega^{\bullet\bullet} + i_{[X,Y]}\omega^{\bullet\bullet} = 0, \end{aligned} \tag{72}$$

where we used (25) and (26), the property of the Lie derivative

$$L_X i_Y \omega = i_Y L_X \omega + i_{[X,Y]}\omega \tag{73}$$

and the relations of the following type:

$$L_{R_E X}\omega = \mathrm{d}i_{R_E X}\omega + i_{R_E X}\,\mathrm{d}\omega = \mathrm{d}i_X\omega^\bullet = L_X\omega^\bullet - i_X\,\mathrm{d}\omega^\bullet = L_X\omega^\bullet. \tag{74}$$

So we proved that for arbitrary vector fields X, Y the contraction of $T(R_E)(X, Y)$ and ω vanishes. But since W bivector is non-degenerate ($W^n \neq 0$), its counter image

$$\omega = \Phi_W^{-1}(W) \tag{75}$$

is also non-degenerate and vanishing of the contraction (72) implies that the torsion $T(R_E)$ itself is zero. So we get

$$\begin{aligned} T(R_E)(X, Y) &= [R_E(X), R_E(Y)] - R_E([R_E(X), Y] \\ &\quad + [X, R_E(Y)] - R_E([X, Y])) = 0. \qquad \square \end{aligned} \tag{76}$$

In summary let us note that the non-Noether symmetries form quite interesting class of symmetries of Hamiltonian dynamical system and lead not only to a number of conservation laws (that under certain conditions ensure integrability), but also enrich the geometry of the phase space by endowing it with several important structures, such as Lax pair, bicomplex, bi-Hamiltonian structure, Frölicher–Nijenhuis operators, etc. The present paper attempts to emphasize deep relationship between different concepts used in construction of conservation laws and non-Noether symmetry. Example of KdV equation suggests that many mysterious objects (for instance, Lenard recursion operator, Lax pair and bi-Hamiltonian structure of KdV equation), that often carry no direct physical content and are considered as purely mathematical constructions resulting the conservation laws, could be regarded as a manifestation of the non-Noether symmetry.

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